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Space and time complexities and sensor threshold selection in quantized identification☆

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ABSTRACT

This work is concerned with system identification of plants using quantized output observations. We focus on relationships between identification space and time complexities. This problem is of importance for system identification in which data-flow rates are limited due to computer networking, communications, wireless channels, etc. Asymptotic efficiency of empirical measure based algorithms yields a tight lower bound on identification accuracy. This bound is employed to derive a separation principle of space and time complexities and to study sensor threshold selection. Insights gained from these understandings provide a feasible approach for optimal utility of communication bandwidth resources in enhancing identification accuracy.

1. Introduction

This paper studies system identification of plants in which only quantized output observations are available. It considers a general class of quantization problems that partition the output range into a finite collection of subsets to achieve identification error reduction. This is of particular interest for systems with a communication channel to transmit output measurements such as sensor networks and networked control systems. In such cases, coding of the output partition can be designed to improve identification accuracy, communication complexity, or other related performance measures.

This subject is important in understanding modeling capability for systems with limited sensor information, establishing relationships between communication resource limitations and identification complexity, and studying sensor networks. This paper is focused on the issues of space and time complexities in their relations to system identification, although the framework can be related to similar problems for state estimation or feedback control. The significance of understanding space and time complexities can be easily illustrated by the following example. For computer information processing of a continuous-time system, its output must be sampled (e.g., with a sampling rate N Hz) and quantized (e.g., with a precision word-length of B bits). Consequently, its output observations carry the data-flow rate of NB bits per second (bps). For instance, for a typical 16-bit precision and 2 kHz sampling rate, a bandwidth 32 K bps of data transmission resource is required, on observations of one output alone. This is a significant resource demand, especially when computer or wireless communications of data are involved.

The problem is generic since any computerized information processing for analog signals will inherently encounter the problem of data precision and sampling rates. However, this problem is not acute when data-flow bandwidths are not limited such as wired systems with fast computers. New technology developments in smart sensors, MEMS (micro-electromechanical systems), sensor networks, computer communication systems, wireless systems, mobile agents, distributed systems, remote controlled systems have ushered in new paradigms in which data-flow rates carry significant costs and limitations.

Conceptually, it is well understood that increasing precision levels is desirable for enhancing accuracy in information

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processing. Similarly, increase in data size can be potentially useful for reducing identification errors. However, these will jointly demand more resources. A fundamental question must be answered: Is such a resource demand necessary for achieving a required identification accuracy? To answer this question, a framework is required that can facilitate analysis of both time complexity (such as the sampling rate) and space complexity (such as the number of subsets for output partition). This paper aims at developing such a framework.

A preliminary study along this direction was initiated in our work Wang, Zhang, and Yin (2003), where a method was introduced for identification of systems with binary-valued output observations, representing the lowest space complexity. The present paper treats more challenging situations of general quantized observations, employing our recent results on asymptotic efficiency of empirical measure based identification algorithms (Wang & Yin, 2007). It addresses the following key issues: (1) What are the main benefits in increasing the space complexity defined by the number of output observation subsets, in terms of identification accuracy? (2) What is the relationship between the space complexity (measurement precision) and time complexity (speed of uncertainty reduction)? (3) How should the output range be partitioned for the best identification accuracy? (4) What is the optimal resource allocation when communication channels provide only limited bandwidths?

Traditional system identification using linear sensors is a relatively mature research area that bears a vast body of literature; see, e.g., the monographs on the subject in a stochastic or worst-case framework (Kushner & Yin, 2003; Ljung, 1987; Milanese & Vicino, 1991). Time complexity in identification was studied in Dahleh, Theodosopoulos, and Tsitsiklis (1993), Poolla and Tikku (1994) and Zames, Lin, and Wang (1994), mostly under deterministic settings. On the other hand, space complexity is a relatively new paradigm in system identification, other than the traditional quantization in which quantization errors are ubiquitously used in analysis. Signal quantization and data compression are a typical A-D conversion process that has been studied extensively in the signal processing and computer science community. Studies of impact of quantization errors can be conducted in a worst-case or probabilistic framework, depending on how quantization errors are modeled. We refer the interested reader to Abut (1990), Gersho and Gray (1992), and Sayood (2000) for a comprehensive coverage of this topic. However, rigorous analysis of space complexity in quantized identification problems ought to take into consideration of general and optimal partition of output ranges. This has not been formulated in the field of identification to the best of our knowledge. Many significant results have been obtained for identification and adaptive control involving random disturbances in the past decades (Chen & Guo, 1991; Hakvoort & Van den Hof, 1995; Kumar, 1990; Kushner & Yin, 2003; Ljung, 1987). The utility of quantized observations carries a flavor that is related to many branches of signal processing problems such as binary reinforcement (Gersho, 1984) and sign-error, sign-regressor, and sign–sign algorithms (Chen & Yin, 2003; Elvitch, Sethares, Rey, & Johnson, 1989; Eweda, 1995). Note that to construct the recursive algorithms, a partial stepsize restarting may be used for certain global optimization problems; see Yin (2000).

The rest of the paper is arranged as follows. Section 2 begins with the problem formulation for system identification using quantized output observations. Early results on identification errors and efficiency of optimal quasi-convex combination estimates (optimal QCCEs) are summarized in Section 3. Section 4 presents asymptotic properties of space and time complexities. These asymptotic properties are independent of specific noise distributions, and hence are fundamental characteristics of quantized identification. Sensor threshold selection and input design are investigated in Section 5. Robust, optimal, and adaptive threshold selections for binary sensors are resolved, for which one may conceptually draw analogy to robust stabilization, optimal performance, and adaptation in control problems. Section 6 is devoted to the study of fundamental tradeoff between space and time complexities in identification problems with constrained communication resources. Optimal resource allocation problems are introduced and solved. Finally, Section 7 concludes the paper with some further remarks.

2. Problem formulation

Consider the system[1]

$$y(k) = \phi^{\mathrm{T}}(k)\theta + d(k), \quad k = 1, 2, \ldots, \tag{1}$$

where $\phi(k) = [u(k), u(k-1), \ldots, u(k-n+1)]^{\mathrm{T}}$ is the vector of the input, $\theta = [a_0, \ldots, a_{n-1}]^{\mathrm{T}}$ is a vector of unknown parameters and $d(k)$ is the disturbance.

The output $y(k)$ is measured by a sensor of m thresholds $-\infty < C_1 < \cdots < C_m < \infty$. The sensor can be represented by a set of m indicator functions

$$s(k) = [s_1(k), \ldots, s_m(k)]^{\mathrm{T}}, \tag{2}$$

where $s_i(k) = I_{\{-\infty < y(k) \le C_i\}}$, $i = 1, \ldots, m$ and

$$I_{\{y(k)\in A\}} = \begin{cases} 1, & \text{if } y(k) \in A, \\ 0, & \text{otherwise.} \end{cases}$$

This representation will be used to derive estimation algorithms.

Assumption A1. Suppose that $\{d(k)\}$ is a sequence of independent and identically distributed (i.i.d.) random variables. The accumulative distribution function $F(\cdot)$ of $d(1)$ is a twice continuously differentiable function. The moment generating function of $d(1)$ exists.

Assumption A2. The prior information on the unknown parameter θ is that $\theta \in \Omega$, where $\Omega \subset \mathbb{R}^n$ (n dimensional space of real values) is a known compact set.

To facilitate analysis, it is convenient to divide the ranges of the sensor into nonoverlapping intervals. An alternative representation of the sensor is by defining $\widetilde{s}_i(k) = I_{\{C_{i-1} < y(k) \le C_i\}}$, $i = 1, \ldots, m+1$ and $\widetilde{s}(k) = \sum_{i=1}^{m+1} i\widetilde{s}_i(k)$. Hence, $\widetilde{s}(k) = i$, for $i = 1, \ldots, m+1$, implies that $y(k) \in (C_{i-1}, C_i]$ with $C_0 = -\infty$ and $C_{m+1} = \infty$ (with the interval (C_m, ∞)).

By using the vector notation, for $l = 1, 2, \ldots$,

$$Y(l) = [y((l-1)n+1), \ldots, y(ln)]^{\mathrm{T}} \in \mathbb{R}^n,$$
$$\Phi(l) = [\phi((l-1)n+1), \ldots, \phi(ln)]^{\mathrm{T}} \in \mathbb{R}^{n\times n},$$
$$D(l) = [d((l-1)n+1), \ldots, d(ln)]^{\mathrm{T}} \in \mathbb{R}^n,$$

the observations can be rewritten in a block form as

$$Y(l) = \Phi(l)\theta + D(l). \tag{3}$$

Note that $\{\Phi(l)\}$ is a sequence of $n \times n$ Toeplitz matrices obtained from the input u.

When the input is designed to be n-periodic and full rank, that is

$$u = \{u_1, u_2, \ldots, u_n, u_1, u_2, \ldots, u_n, \ldots\}$$

[1] Extension of the results from FIR systems to rational systems can be achieved by using the methods in Wang, Yin, and Zhang (2006).

and

$$\Phi(l) = \Phi = \begin{bmatrix} u_n & u_{n-1} & \cdots & u_1 \\ u_1 & u_n & & u_2 \\ & & \ddots & \\ u_{n-1} & u_{n-2} & \cdots & u_n \end{bmatrix}$$

is full rank, we have $Y(l) = \Phi\theta + D(l)$. Let

$$\zeta = [\zeta_1, \ldots, \zeta_n]^{\mathrm{T}} = \Phi\theta$$

and

$$S_i(l) = I_{\{Y(l) \le C_i \mathbb{1}_n\}} = I_{\{D(l) \le C_i \mathbb{1}_n - \zeta\}},$$

where $\mathbb{1}_n$ is the n dimensional vector whose elements are all equal to 1. For $i = 1, \ldots, m+1$ and $j = 1, \ldots, n$, denote $S_{i,j}(l) = I_{\{d((l-1)n+j) \le C_i - \zeta_j\}}$. Then

$$S_i(l) = [S_{i,1}(l), \ldots, S_{i,n}(l)]^{\mathrm{T}}.$$

3. Identification algorithms and optimality

We briefly summarize certain key results that form the basis for complexity analysis, threshold selection, and input design. The details of this section can be found in Wang and Yin (2007).

For the system in (1), the probability of $\{S_{i,j}(l) = 1\}$ is

$$p_{i,j} = P\{d((l-1)n + j) \le C_i - \zeta_j\} = F(C_i - \zeta_j), \tag{4}$$

which is independent of l or n since d is i.i.d. Take N measurements on $S_{i,j}(l)$. Then, for $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$,

$$\varphi_{i,j}(N) = \frac{1}{N} \sum_{l=1}^{N} S_{i,j}(l) \tag{5}$$

is the sample relative frequency of $d(k)$ taking values in $(-\infty, C_i - \zeta_j]$. Thus, by the strong law of large numbers,

$$\varphi_{i,j}(N) \to p_{i,j} = F(C_i - \zeta_j), \quad \text{w.p.1.} \tag{6}$$

Note that F is a monotone function by Assumption A1, and Ω is bounded by Assumption A2. Then, there exists a small $z > 0$ such that

$$z < p_{i,j} = F(C_i - \zeta_j) < 1 - z.$$

Since $F(\cdot)$ is not invertible at 0 and 1, we modify $\varphi_{i,j}$ to avoid singularities. Let

$$\xi_{i,j}(N) = \begin{cases} \varphi_{i,j}(N), & \text{if } z \le \varphi_{i,j}(N) \le 1 - z; \\ z, & \text{if } \varphi_{i,j}(N) < z; \\ 1 - z, & \text{if } \varphi_{i,j}(N) > 1 - z. \end{cases} \tag{7}$$

Since $\varphi_{i,j}(N) \to p_{i,j}$, w.p.1 and $z < p_{i,j} < 1 - z$, we have $\xi_{i,j}(N) \to p_{i,j}$, w.p.1. Denote

$$\xi_i(N) = [\xi_{i,1}(N), \ldots, \xi_{i,n}(N)]^{\mathrm{T}}. \tag{8}$$

By Assumption A1, F has a continuous inverse. Hence, for each $i = 1, \ldots, n$,

$$\begin{aligned} \zeta_i(N) &= [\zeta_{i,1}(N), \ldots, \zeta_{i,n}(N)]^{\mathrm{T}} \\ &:= C_i \mathbb{1}_n - \mathbf{F}^{-1}(\xi_i(N)) \\ &\to C_i \mathbb{1}_n - \mathbf{F}^{-1}(p_i) = \zeta \quad \text{w.p.1,} \end{aligned} \tag{9}$$

where $p_i = [p_{i,1}, \ldots, p_{i,n}]^{\mathrm{T}}$.

Let $\zeta_{i,j}(N)$ $(i = 1, \ldots, m)$ be m estimators of ζ_j based on samples of size N. Denote the estimation errors by $e_{i,j}(N) = \zeta_{i,j}(N) - \zeta_j$, $\zeta_{*,j} = [\zeta_{1,j}, \ldots, \zeta_{m,j}]^{\mathrm{T}}$ and $e_{*,j}(N) = [e_{1,j}(N), \ldots, e_{m,j}(N)]^{\mathrm{T}}$. Then, $e_{*,j}(N) = \zeta_{*,j}(N) - \zeta_j \mathbb{1}_m$, and for $\gamma_j = [\gamma_{j,1}, \ldots, \gamma_{j,m}]^{\mathrm{T}}$, $j = 1, \ldots, n$, satisfying $\gamma_{j,1} + \cdots + \gamma_{j,m} = 1$,

$$\widehat{\zeta}_j(N) = \sum_{i=1}^{m} \gamma_{j,i} \zeta_{i,j}(N) = \gamma_j^{\mathrm{T}} \zeta_{*,j}(N) \tag{10}$$

is called a *Quasi-Convex Combination Estimator QCCE*. Let

$$\widehat{\zeta} = [\widehat{\zeta}_1, \ldots, \widehat{\zeta}_n]^{\mathrm{T}}.$$

Then, an estimator of θ can be constructed as

$$\widehat{\theta}(N) = \Phi^{-1} \widehat{\zeta}(N). \tag{11}$$

$\widehat{\zeta}_j(N)$ is an asymptotically unbiased estimator of ζ_j (Wang & Yin, 2007; Zhao, Zhang, Wang, & Yin, submitted for publication). The variance of the estimation error $\widehat{\zeta}_j(N) - \zeta_j$ is given by

$$\overline{\sigma}_j^2(N) = \gamma_j^{\mathrm{T}} V_j(N) \gamma_j,$$

where

$$V_j(N) = E e_{*,j}(N) e_{*,j}^{\mathrm{T}}(N). \tag{12}$$

The estimator that minimizes $\overline{\sigma}^2(N)$ is called *the optimal quasi-convex combination estimator (optimal QCCE)*, which is obtained from

$$\sigma_j^2(N) = \min_{\gamma_j, \gamma_j^{\mathrm{T}} \mathbb{1} = 1} \overline{\sigma}_j^2(N) = \min_{\gamma_j, \gamma_j^{\mathrm{T}} \mathbb{1} = 1} \gamma_j^{\mathrm{T}} V_j(N) \gamma_j. \tag{13}$$

Theorem 1 (*Wang & Yin, 2007*). *Suppose that Assumption A1 holds and $V_j(N)$ is positive definite. Then the optimal QCCE can be obtained by choosing*

$$\gamma_j^* = \frac{V_j^{-1}(N)\mathbb{1}}{\mathbb{1}^{\mathrm{T}} V_j^{-1}(N) \mathbb{1}}, \qquad \widehat{\zeta}_j(N) = (\gamma_j^*)^{\mathrm{T}} \zeta_{*,j}(N), \tag{14}$$

and the minimal variance is

$$\sigma_j^2(N) = \frac{1}{\mathbb{1}^{\mathrm{T}} V_j^{-1}(N) \mathbb{1}}. \tag{15}$$

Theorem 2 (*Wang & Yin, 2007;Zhao et al., submitted for publication*). *Under Assumption A1,*

$$R_j(N) := N V_j(N) \to \Lambda_j W_j \Lambda_j := R_j, \quad N \to \infty, \tag{16}$$

where $\Lambda_j = \mathrm{diag}\{\frac{1}{f(C_1 - \zeta_j)}, \ldots, \frac{1}{f(C_m - \zeta_j)}\}$ and

$$W_j = \begin{pmatrix} p_{1,j}(1 - p_{1,j}) & \cdots & p_{1,j}(1 - p_{l,j}) \\ \vdots & \ddots & \vdots \\ p_{1,j}(1 - p_{l,j}) & \cdots & p_{l,j}(1 - p_{l,j}) \end{pmatrix}. \tag{17}$$

To implement the optimal QCCE algorithm, one may replace C_i, ζ_j, and $p_{i,j}$ in Theorem 2 by their estimates, to obtain $R_j(N) = \Lambda_j(N) W_j(N) \Lambda_j(N)$ and

$$\gamma_j^* = \frac{R_j^{-1}(N)\mathbb{1}}{\mathbb{1}^{\mathrm{T}} R_j^{-1}(N) \mathbb{1}}.$$

Recall that $S_{i,j}(l) = I_{\{d(k) \le C_i\}}$. Denote $\widetilde{S}_{i,j}(l) = I_{\{C_{i-1} - \zeta_j < d((l-1)n+j) \le C_i - \zeta_j\}}$, $i = 1, \ldots, m+1$, $j = 1, \ldots, n$, with $C_0 = -\infty$ and $C_{m+1} = \infty$. Let

$$\begin{aligned} \widetilde{p}_{i,j} &= P\{C_{i-1} - \zeta_j < d(1) \le C_i - \zeta_j\} \\ &= F(C_i - \zeta_j) - F(C_{i-1} - \zeta_j) := \widetilde{F}_{i,j}. \end{aligned}$$

Define

$$\widetilde{h}_{i,j} = \partial \widetilde{F}_{i,j} / \partial \zeta_j = -f(C_i - \zeta_j) + f(C_{i-1} - \zeta_j).$$

Lemma 1 (*Wang & Yin, 2007*). *The Cramér–Rao lower bound for estimating ζ_j based on observations of $\{s(k)\}$ is*

$$\sigma^2_{j,CR}(N, m, \theta) = \left(N \sum_{i=1}^{m+1} \frac{\widetilde{h}^2_{i,j}}{\widetilde{p}_{i,j}} \right)^{-1}. \tag{18}$$

With $\theta = \Phi^{-1}\zeta$, we obtain the following bounds.

Theorem 3. *The Cramér–Rao lower bound for estimating θ based on observations of $\{s(k)\}$ is*

$$\Sigma_{CR}(N, m, \theta) = \Phi^{-1} \Sigma_{CR}(N, m, \zeta)(\Phi^{\mathrm{T}})^{-1}, \tag{19}$$

where

$$\Sigma_{CR}(N, m, \zeta) = \mathrm{diag}^{-1} \left\{ N \sum_{i=1}^{m+1} \frac{\widetilde{h}^2_{i,1}}{\widetilde{p}_{i,1}}, \ldots, N \sum_{i=1}^{m+1} \frac{\widetilde{h}^2_{i,n}}{\widetilde{p}_{i,n}} \right\}. \tag{20}$$

The following theorem reveals that the optimal QCCE is asymptotically efficient.

Theorem 4. *The optimal QCCE is asymptotically efficient in the sense that*

$$N\Sigma(N, m, \theta) - N\Sigma_{CR}(N, m, \theta) \to 0 \quad \text{as } N \to \infty.$$

In what follows, $\Sigma^2_{CR}(N, m, \theta)$ will be used to guide complexity analysis and threshold selection.

4. Space and time complexities

The number m of thresholds is a measure of space complexity, whereas the observation length N is a measure of time complexity that quantifies how fast uncertainty can be reduced. In an information processing problem that involves computer networks, or communication data transmission, the required resource is usually represented by bandwidths in bits per second. In our identification problems, if identification must be accomplished in T seconds to facilitate subsequent tasks (control, prediction, diagnosis, etc.), then the time complexity N is translated to N samples per T seconds. Correspondingly, the required bandwidth will be $R = N \log(m + 1)$ (the simplified notation $\log = \log_2$ is used) bits per T seconds, or R/T bps. Since T is an external constant, we shall simply view R as the required bandwidth on communication channels. For an available total resource R, one may choose to assign more resource to space complexity (increasing m) or to time complexity (increasing N, i.e., the rate of data acquisition). The overall goal is to achieve the best uncertainty reduction for a given resource, or to achieve minimum resource utilization for a given level of uncertainty reduction. This section presents basic properties of time and space complexities.

4.1. Separation of time and space complexities

To understand the impact of sensor threshold values and the number of thresholds on identification error variance, consider $\Sigma_{CR}(N, m, \theta)$ in (19). We will interpret exchangeably m as space complexity, the number of cascaded binary-valued sensors, or the number of sensor thresholds. Let us first fix an integer m. $\Sigma_{CR}(N, m, \theta)$ indicates a basic relationship that delineates a fundamental property of asymptotic separation of space and time complexity in variance reduction.

Corollary 1.

$$\Sigma_0(m, \theta) := N\Sigma_{CR}(N, m, \theta) \tag{21}$$

is independent of N.

Define

$$\varrho(m, \theta) = \sup_{\|x\|=1} \|\Sigma_0(m, \theta)x\|, \tag{22}$$

namely, the largest singular value of $\Sigma_0(m, \theta)$.

Remark 1. Corollary 1 shows that asymptotically the optimal variance $\sigma^2_{CR}(N, m, \theta)$ is reduced in the rate of $1/N$ in terms of its time complexity. Its reduction by increasing space complexity m is characterized entirely by $\varrho(m, \theta)$, which is independent of N. This separation of space and time complexity in their capability for identification error reduction provides a convenient foundation for complexity analysis. Consequently, $\varrho(m, \theta)$ will be used for analysis of space complexity.

4.2. Monotonicity of space complexity

Let $[y_{\min}, y_{\max}]$ be the range of y_k, which may be unbounded. A placement of m sensor thresholds is a partition $y_{\min} < C_1 < \cdots < C_m < y_{\max}$ of the interval $[y_{\min}, y_{\max}]$. In what follows, we also use the notation $\mathcal{S}_m = \{y_{\min}, C_1, \ldots, C_m, y_{\max}\}$ to denote the set of points of the partition.

Definition 1. Suppose $m_1 < m_2$ are two positive integers, and $\mathcal{S}_{m_1} = \{y_{\min}, C_1, \ldots, C_{m_1}, y_{\max}\}$ and $\mathcal{S}_{m_2} = \{y_{\min}, C^*_1, \ldots, C^*_{m_2}, y_{\max}\}$ are two placements of sensors. We say that $\mathcal{S}_{m_2}$ is a *refinement* of $\mathcal{S}_{m_1}$, if $\{y_{\min}, C_1, \ldots, C_{m_1}, y_{\max}\}$ is a subset of $\{y_{\min}, C^*_1, \ldots, C^*_{m_2}, y_{\max}\}$.

Remark 2. In the definition of placement of sensor thresholds, $[y_{\min}, y_{\max}]$ can be either finite or infinite. In case one of these values is ∞ or $-\infty$, it is understood that we work with the extended real number system. For practical utility, we have assumed that no threshold is placed at either $y_{\min}$ or $y_{\max}$, otherwise they do not provide any useful information for system identification.

Note that the statement "$\mathcal{S}_{m_2}$ is a refinement of $\mathcal{S}_{m_1}$" means that $\mathcal{S}_{m_2}$ can be obtained by starting with the threshold points $C_1 < \cdots < C_{m_1}$ and interposing $m_2 - m_1$ points between them to form a finer subdivision.

Theorem 5. *Suppose that $\mathcal{S}_{m_1}$ and $\mathcal{S}_{m_2}$ are two placements of sensor thresholds such that $\mathcal{S}_{m_2}$ is a refinement of $\mathcal{S}_{m_1}$. Then $\varrho(m, \theta)$ given in (22) satisfies*

$$\varrho(m_2, \theta) \le \varrho(m_1, \theta).$$

Remark 3. Theorem 5 is a result about monotonicity of space complexity. It indicates that a reduction of error variance is achieved by increasing space complexity.

Proof. The extra thresholds in $\mathcal{S}_{m_2}$ that are not in $\mathcal{S}_{m_1}$ can be added one at a time. Hence, we only need to show that if one additional threshold C is added to $\mathcal{S}_{m_1}$, we have $\varrho(m_1 + 1, \theta) \le \varrho(m_1, \theta)$. Since the following analysis is valid for each element of $\Sigma_{CR}(m, \theta)$, we shall concentrate on a generic element σ^2_{CR} for analysis.

Suppose that one additional threshold C is inserted in $\mathcal{S}_{m_1}$, between $C_1, C_2 \in \mathcal{S}_{m_1}$, $C_1 < C < C_2$. Denote $p = P\{C_1 < x \le C_2\}$, $p_1 = P\{C_1 < x \le C\}$, $p_2 = P\{C < x \le C_2\}$ and $h = \frac{\partial p}{\partial x}$, $h_1 = \frac{\partial p_1}{\partial x}$, $h_2 = \frac{\partial p_2}{\partial x}$. Note that $p = p_1 + p_2$ and $h = h_1 + h_2$, we have

$$\frac{h_1^2}{p_1} + \frac{(h - h_1)^2}{p - p_1} - \frac{h^2}{p} = \frac{(h_1 p - h p_1)^2}{p_1(p - p_1)p} \ge 0,$$

which implies

$$\frac{h_1^2}{p_1}+\frac{h_2^2}{p_2}\geq\frac{h^2}{p}.$$

By (20), we obtain

$$\sigma_{CR}^2(N,m_1+1,\zeta)\leq\sigma_{CR}^2(N,m_1,\zeta),$$

and hence, by (19),

$$\sigma_{CR}^2(N,m_1+1,\theta)\leq\sigma_{CR}^2(N,m_1,\theta).$$

Thus, $\varrho(m_1+1,\theta)\leq\varrho(m_1,\theta)$ owing to (21) and (22). □

For simplicity, let $u=\{u_0,0,\ldots,0,u_0,0,\ldots,0,u_0,0,\ldots\}$. Without loss of generality, assume $u_0>0$. In this case,

$$\Phi=\mathrm{diag}\{u_0,\ldots,u_0\}\quad\text{and}\quad\zeta=u_0\theta.$$

By (19) and (22), we have

$$\Sigma_{CR}(N,m,\theta)=u_0^{-2}\mathrm{diag}^{-1}\left\{N\sum_{i=1}^{m+1}\frac{\widetilde{h}_{i,1}^2}{\widetilde{p}_{i,1}},\ldots,N\sum_{i=1}^{m+1}\frac{\widetilde{h}_{i,n}^2}{\widetilde{p}_{i,n}}\right\}.$$

Similar to (22), define

$$\eta(m,\theta)=\max_j\left\{u_0^{-2}\left(\sum_{i=1}^{m+1}\frac{\widetilde{h}_{i,j}^2}{\widetilde{p}_{i,j}}\right)^{-1}\right\}. \tag{23}$$

We will show that $\eta(m,\theta)$ is useful for threshold selection, input design, and characterization of time and space complexities.

5. Threshold selection for binary-valued sensors

We now consider the problem of threshold selection. An interval $\mathcal{I}_i=(C_{i-1},C_i]$ of the output range can provide useful information for system identification only when $\widetilde{h}_{i,j}\neq 0,\forall j$. Contribution of a sensor interval to error reduction depends on the actual parameter θ, the distribution function F, the thresholds, and the input, as illustrated by the following two examples.

Example 1. Suppose that $F(\cdot)$ is a uniform distribution on [0, 10], namely $F(x)=x/10$, $0\leq x\leq 10$. The prior information on θ is that $\theta\in[2,5]$. If one selects $u(k)\equiv u_0=1$, and places four sensor thresholds at $C_1=1$, $C_2=6$, $C_3=10$, and $C_4=20$, then it can be verified that $F(C_1-\theta)=0$, $F(C_2-\theta)=(6-\theta)/10$, $F(C_3-\theta)=(10-\theta)/10$, $F(C_4-\theta)=1$. These imply

$$\widetilde{p}_1=0,\qquad\widetilde{p}_2=(6-\theta)/10,\qquad\widetilde{p}_3=0.4,\qquad\widetilde{p}_4=\theta/10.$$

Since $\widetilde{p}_1$ and $\widetilde{p}_3$ do not depend on θ, the intervals $(-\infty,C_1]$ and $(C_2,C_3]$ do not provide information about θ.

Even when the threshold does provide information, selection of the threshold value will have significant impact on the convergence speed.

Example 2. Suppose that $F(\cdot)$ is a Gaussian distribution with zero mean and variance $\sigma^2=625$. The true parameter is $\theta=100$ and $u\equiv 1$. In this case, Corollary 1 becomes, adding C in notation and using $\eta_C(m,\theta)$ in place of $\Sigma_0(m,\theta)$,

$$\eta_C(1,100)=N\sigma_{CR}^2(1,N,\theta)=\left(\frac{\widetilde{h}_1^2}{\widetilde{p}_1}+\frac{\widetilde{h}_2^2}{\widetilde{p}_2}\right)^{-1}=\frac{F(C-\theta)(1-F(C-\theta))}{f^2(C-\theta)}.$$

Since $F(\cdot)$ is Gaussian, its support is $(-\infty,\infty)$, so for $C=20,50,80,100$, each threshold provides certain information. For each C, we obtain the values

$$\eta_{20}(1,100)=75\,506;\qquad\eta_{50}(1,100)=4767;$$
$$\eta_{80}(1,100)=1244;\qquad\eta_{100}(1,100)=982.$$

These indicate that the convergence speed η_C is significantly impacted by the values of the threshold.

5.1. Feasibility analysis

In an identification problem, the parameter $\theta=[a_1,\ldots,a_n]^{\mathrm{T}}$ is unknown. Hence, one must work with the prior uncertainty set on θ. While in most applications, $a_j\in[\underline{a},\overline{a}]$ $(j=1,\ldots,n)$ are the typical prior information, we shall use the general prior uncertainty set $\theta\in\Omega$ to include other possibilities. We first concentrate on a binary-valued sensor of threshold C. Similar conclusions will later be derived for general quantized sensors. It is assumed that C and u_0 can be either selected prior to an identification experiment, or tuned during it. Let $\widetilde{p}_j=F(C-a_ju_0)$ and $\widetilde{h}_j=\partial\widetilde{p}_j/\partial a_j=-f(C-a_ju_0)u_0$.

Definition 2. An interval $\mathcal{I}=(-\infty,C]$ or a threshold C is said to be (1) *feasible* for $\theta=[a_1,\ldots,a_n]^{\mathrm{T}}$ if the corresponding $\widetilde{h}_j\neq 0$, $j=1,\ldots,n$; (2) *robustly feasible* for Ω, if it is feasible for all $\theta\in\Omega$.

For a given Ω, the set of all feasible thresholds will be denoted by Γ_Ω. We will derive concretely Γ_Ω for some typical cases.

For a given $\theta=[a_1,\ldots,a_n]^{\mathrm{T}}$, denote $\theta_{\max}=\max_{j=1,\ldots,n}a_j$ and $\theta_{\min}=\min_{j=1,\ldots,n}a_j$. For a set Ω_0, let

$$\overline{a}=\sup_{\theta\in\Omega_0}\theta_{\max}\quad\text{and}\quad\underline{a}=\inf_{\theta\in\Omega_0}\theta_{\min}$$

Theorem 6. *Suppose that the prior information on the unknown parameters is $\theta\in\Omega_0$, and the disturbance $d(k)$ is zero mean and its density function has support (i.e., strictly positive) in $(-\delta,\delta)$. For a given $u_0>0$, the set of robustly feasible thresholds for Ω_0 is*

$$\Gamma_{\Omega_0}=\{C:\overline{a}u_0-\delta<C<\underline{a}u_0+\delta\}. \tag{24}$$

Furthermore, Γ_{Ω_0} is non-empty if and only if

$$\delta>\frac{(\overline{a}-\underline{a})u_0}{2}. \tag{25}$$

Proof. For $\forall C\in\Gamma_{\Omega_0}$, we have $C>\overline{a}u_0-\delta\geq a_ju_0-\delta$ and $C<\underline{a}u_0-\delta\leq a_ju_0-\delta$, namely, $a_ju_0-\delta<C<a_ju_0+\delta$ for all $\theta\in\Omega_0$ and $j=1,\ldots,n$, that is, $-\delta<C-a_ju_0<\delta$. Hence $\widetilde{h}_j=\partial\widetilde{p}_j/\partial a_j=-f(C-a_ju_0)u_0\neq 0$. By definition, C is robustly feasible for Ω_0. On the other hand, if $C\notin\Gamma_{\Omega_0}$, then there exists some $\theta\in\Omega_0$ which has at least one parameter a_j such that $C\notin(a_ju_0-\delta,a_ju_0+\delta)$, which implies C is not feasible for θ. Therefore, C is not robustly feasible for Ω_0. Thus, the first part of the theorem is proved.

Note that Γ_{Ω_0} is non-empty if and only if $\overline{a}u_0-\delta<\underline{a}u_0+\delta$, which is equivalent to (25). So the second part is true. □

Remark 4. Note that (25) can be rewritten as

$$u_0<\frac{2\delta}{\overline{a}-\underline{a}}, \tag{26}$$

which defines the maximum input value.

5.2. Worst-case optimal design

For a given prior uncertainty set Ω of the unknown parameter θ, the set Γ_Ω of robustly feasible thresholds can be used to select the thresholds to reduce identification errors. Corollary 1 provides the main vehicle for this pursuit.

Observe that

$$\widetilde{p}_{1,j} = F(C - a_j u_0), \qquad \widetilde{p}_{2,j} = 1 - F(C - a_j u_0),$$
$$\widetilde{h}_{1,j} = -f(C - a_j u_0), \qquad \widetilde{h}_{2,j} = f(C - a_j u_0).$$

By adding the dependence on C and u_0 in notation, (22) is reduced to

$$\eta_{C,u_0}(1,\theta) = \max_j \left\{ u_0^{-2} \left(\frac{\widetilde{h}_{1,j}^2}{\widetilde{p}_1} + \frac{\widetilde{h}_{2,j}^2}{\widetilde{p}_{2,j}} \right)^{-1} \right\} = \max_j \frac{F(C - a_j u_0)(1 - F(C - a_j u_0))}{f^2(C - a_j u_0) u_0^2}.$$

If $\theta \in \Omega$ and $C \in \Gamma_\Omega$, then $f(C - a_j u_0) \neq 0$ for $j = 1, \ldots, n$. Therefore, $\eta_{C,u_0}(1,\theta)$ is well defined. Denote

$$G(x) = F(x)(1 - F(x))/f^2(x).$$

Then,

$$\eta_{C,u_0}(1,\theta) = \max_j G(C - a_j u_0)/u_0^2. \tag{27}$$

In order to minimize $\eta_{C,u_0}(1,\theta)$, we may choose optimal C first for a given u_0, followed by an optimization on u_0; or choose optimal u_0 for a given C and then optimize C. For the first case, by (27) the optimization on C is to minimize $\max_j G(C - a_j u_0)$. In contrast, for a given C, u_0 affects not only $G(x)$ but also the denominator u_0^2, which makes the problem more difficult. For this reason, we opt for the min–max problem

$$\eta^* = \inf_{u_0 > 0} \inf_{C \in \Gamma_\Omega} \sup_{\theta \in \Omega} \eta_{C,u_0}(1,\theta).$$

We solve this problem more concretely under certain typical situations.

Bounded disturbances. Suppose that the prior information on the unknown parameter is $\theta \in \Omega_0 = \{\theta = [a_1, \ldots, a_n]^{\mathrm{T}} : a_j \in [\underline{a}, \overline{a}], j = 1, \ldots, n\}$, and the disturbance $d(k)$ is zero mean and its density function has support on $(-\delta, \delta)$. By Theorem 6, the set of robustly feasible thresholds for Ω_0 is $\Gamma_{\Omega_0} = \{C : \overline{a}u_0 - \delta < C < \underline{a}u_0 + \delta\}$. The optimal threshold and input selection is obtained by solving the following min–max optimization problem

$$\eta^* = \inf_{u_0 > 0} \inf_{C \in \Gamma_{\Omega_0}} \sup_{\theta \in \Omega_0} \max_j G(C - a_j u_0)/u_0^2. \tag{28}$$

Theorem 7. *Suppose that $d(k)$ is uniformly distributed with density function $f(x) = 1/(2\delta)$ for $x \in (-\delta, \delta)$.*

(i) *If $(\overline{a} - \underline{a})u_0/2 < \delta \leq (\overline{a} - \underline{a})u_0$, then*

$$\eta^* = \frac{(\overline{a} - \underline{a})^2}{4} \tag{29}$$

and any $C \in (\overline{a}u_0 - \delta, \underline{a}u_0 + \delta)$ is optimal in the worst-case sense over $\theta \in \Omega_0$.

(ii) *If $\delta > (\overline{a} - \underline{a})u_0$, then*

$$\eta^* = (\overline{a} - \underline{a})^2. \tag{30}$$

Proof. For a given $u_0 > 0$, and for each $C \in \Gamma_{\Omega_0}$ and $\theta \in \Omega_0$, we have

$$C - a_j u_0 < \underline{a}u_0 + \delta - \underline{a}u_0 = \delta,$$

and

$$C - a_j u_0 > \overline{a}u_0 - \delta - \overline{a} = -\delta.$$

It follows that in (28), $f(C - a_j u_0) = 1/(2\delta)$ and $F(C - a_j u_0) = (C - a_j u_0 + \delta)/(2\delta)$. As a result,

$$G(C - a_j u_0)/u_0^2 = (2\delta)^2 \frac{C - a_j u_0 + \delta}{2\delta u_0^2} \left(1 - \frac{C - a_j u_0 + \delta}{2\delta}\right) = \frac{\delta^2 - (C - a_j u_0)^2}{u_0^2}.$$

(i) In the case of $(\overline{a} - \underline{a})u_0/2 < \delta \leq (\overline{a} - \underline{a})u_0$, we have

$$\underline{a}u_0 \leq \overline{a}u_0 - \delta; \qquad \underline{a}u_0 + \delta \leq \overline{a}u_0.$$

This implies that for any $C \in (\overline{a}u_0 - \delta, \underline{a}u_0 + \delta)$, there exists $\theta \in \Omega_0$ such that $a_j u_0 = C$. Consequently, for any given $C \in \Gamma_{\Omega_0}$,

$$\max_j \frac{\delta^2 - (C - a_j u_0)^2}{u_0^2} = \frac{\delta^2}{u_0^2}.$$

This and (26) yield that

$$\eta^* = \inf_{u_0 > 0} \inf_{C \in \Gamma_\Omega} \max_j G(C - a_j u_0)/u_0^2 = \inf_{u_0 < 2\delta/(\overline{a} - \underline{a})} \frac{\delta^2}{u_0^2} = \frac{(\overline{a} - \underline{a})^2}{4}.$$

Thus, (29) is proved.

(ii) In case of $\delta > (\overline{a} - \underline{a})u_0$, we have $u_0 < \delta/(\overline{a} - \underline{a})$ and $[\underline{a}u_0, \overline{a}u_0] \subset (\overline{a}u_0 - \delta, \underline{a}u_0 + \delta)$. For $C \in (\overline{a}u_0 - \delta, \underline{a}u_0 + \delta)$ but $C \notin [\underline{a}u_0, \overline{a}u_0]$, we have that if $C > \overline{a}u_0$

$$\max_{\theta \in \Omega_0} \max_j \frac{\delta^2 - (C - a_j u_0)^2}{u_0^2} = \frac{\delta^2 - \min_{\theta \in \Omega_0} (C - \overline{a}_j u_0)^2}{u_0^2} = \frac{\delta^2 - (C - \overline{a}u_0)^2}{u_0^2},$$

or if $C < \underline{a}u_0$,

$$\max_{\theta \in \Omega_0} \max_j \frac{\delta^2 - (C - \theta u_0)^2}{u_0^2} = \frac{\delta^2 - (\underline{a}u_0 - C)^2}{u_0^2}.$$

In the first case, the variance is minimized when C is closest to $\underline{a}u_0 + \delta$, and

$$\eta^* = \inf_{u_0 < \delta/(\overline{a} - \underline{a})} \frac{\delta^2 - (\underline{a}u_0 + \delta - \overline{a}u_0)^2}{u_0^2} = \inf_{u_0 < \delta/(\overline{a} - \underline{a})} \frac{2\delta(\overline{a} - \underline{a})u_0 - (\overline{a} - \underline{a})^2 u_0^2}{u_0^2} = \inf_{u_0 < \delta/(\overline{a} - \underline{a})} \frac{2\delta(\overline{a} - \underline{a})}{u_0} - (\overline{a} - \underline{a})^2 = (\overline{a} - \underline{a})^2.$$

In the second case, the variance is minimized when C is closest to $\overline{a}u_0 - \delta$ and similarly, we have $\eta^* = (\overline{a} - \underline{a})^2$. Thus, (30) is proved. □

Unbounded disturbances. When the disturbance is unbounded such that $f(x) > 0$ for all x, for any given u_0, all $C \in \mathbb{R}$ are robustly feasible for Ω. The optimal threshold selection becomes

$$\eta^* = \inf_{u_0 > 0} \inf_C \max_{\theta \in \Omega} \max_j G(C - a_j u_0) \Big/ u_0^2. \tag{31}$$

The solutions to (31) can be obtained by first calculating

$$\widetilde{\eta}(C,u_0)=\max_{\theta\in\Omega}\max_j G(C-a_ju_0)\Big/u_0^2,\quad C\in\mathbb{R},\ u_0>0,$$

and then,

$$\eta^*=\inf_{u_0>0}\inf_C \widetilde{\eta}(C,u_0).$$

Suppose $d(k)$ is Gaussian distributed with mean 0 and variance σ^2. Denote the corresponding density and distribution functions by $f_\sigma(x)$ and $F_\sigma(x)$, respectively. Let $G_\sigma(x)=F_\sigma(x)(1-F_\sigma(x))/f_\sigma^2(x)$ and $H_\sigma(x)=G_\sigma(x)/x^2$.

Theorem 8. *$G_\sigma(x)$ and $H_\sigma(x)$ have the following properties.*

(i) *$G'_\sigma(x)=\mathrm{d}G_\sigma(x)/\mathrm{d}x>0$ for $x>0$ and $G'_\sigma(x)<0$ for $x<0$.*
(ii) *There exists a unique $x^*_\sigma\in(0,\infty)$ such that $H_\sigma(x^*_\sigma)=\min_{x>0}H_\sigma(x)$, where $x^*_\sigma=\sigma x^*_1$ and x^*_1 is the unique solution of*

$$h(x)=\left(x-\frac{1}{x}\right)\cdot\frac{2F_1(x)}{2F_1(x)-1}-\frac{f_1(x)}{1-F_1(x)}=0. \tag{32}$$

Proof. (i) By Gaussian distribution, $G_\sigma(-x)=G_\sigma(x)$, and hence we need only to prove the case $x>0$. Since $F_\sigma(x)=F_1(x/\sigma)$ and $f_\sigma(x)=\sigma f_1(x/\sigma)$, we have $G_\sigma(x)=\sigma^2G_1(x/\sigma)$ and $G'_\sigma(x)=\sigma G'_1(x/\sigma)$. As a result, $G'_\sigma(x)>0$ for all $x>0$ if and only if $G'_1(x)>0$ for all $x>0$. Without loss of generality, let $\sigma=1$.

Denote

$$\begin{aligned}
g_1(x)&=f_1(x)-x(1-F_1(x)),\\
g_2(x)&=xf_1(x)-(1+x^2)(1-F_1(x)),\\
g_3(x)&=(1-F_1(x))^2+xf_1(x)(1-F_1(x))-f_1^2(x),\\
g_4(x)&=2(f_1(x)+xF_1(x))-f_1(x)/(1-F_1(x)).
\end{aligned}$$

Since $\lim_{x\to\infty}g_1(x)=0$ and

$$\begin{aligned}
g'_1(x)&=-xf_1(x)-(1-F_1(x))+xf_1(x)\\
&=-(1-F_1(x))<0,
\end{aligned}$$

$g_1(x)>0$. Similarly, from $\lim_{x\to\infty}g_2(x)=0$ and $g'_2(x)=2g_1(x)>0$, we have $g_2(x)<0$. Also $\lim_{x\to\infty}g_3(x)\to 0$ and $g'_3(x)=f_1(x)g_2(x)<0$ imply $g_3(x)>0$. Hence

$$\begin{aligned}
g'_4(x)&=2F_1(x)-\frac{-xf_1(x)(1-F_1(x))+f_1^2(x)}{(1-F_1(x))^2}\\
&>2F_1(x)-1>0.
\end{aligned}$$

Since $F_1(0)=0.5$, we have $g_4(0)=0$ and $g_4(x)>0$ for $x>0$. This implies $G'_1(x)=g_4(x)(1-F_1(x))/f_1^2(x)>0$ for $x>0$.

(ii) Since $H_\sigma(x)=G_\sigma(x)/x^2=H_1(x/\sigma)$, we have $\min_x H_\sigma(x)=\min_x H_1(x/\sigma)=\min_x H_1(x)$. If x^*_1 is the unique point such that $H_1(x^*_1)=\min_x H_1(x)$, then $x^*_\sigma=\sigma x^*_1$ satisfies

$$\begin{aligned}
H_\sigma(x^*_\sigma)&=H_\sigma(\sigma x^*_1)=H_1(x^*_1)\\
&=\min_x H_1(x)=\min_x H_\sigma(x).
\end{aligned}$$

Therefore, we need only prove that $H'_1(x)=0$ has a unique solution in $(0,\infty)$. Since $\lim_{x\to0+}H_1(x)=\lim_{x\to\infty}H_1(x)=\infty$, there exists $x\in(0,\infty)$ such that $H'_1(x)=0$ by continuity. $h(x)$ and $H'_1(x)$ are related by

$$h(x)=H'_1(x)f_1^4(x)x^3/(2F_1(x)-1)(1-F_1(x))$$

which implies that $H'_1(x)=0$ if and only if $h(x)=0$ for $x>0$. Denote

$$\begin{aligned}
h_1(x)&=2F_1(x)-1-2xf_1(x),\\
h_2(x)&=\left(x-\frac{1}{x}\right)\frac{2F_1(x)}{2F_1(x)-1},\\
h_3(x)&=\frac{f_1(x)}{1-F_1(x)}.
\end{aligned}$$

Then, $h(x)=h_2(x)-h_3(x)$. Since $h_1(0)=0$ and $h'_1(x)=2x^2f_1(x)>0$, $h_1(x)>0$. Furthermore,

$$\begin{aligned}
h'_2(x)&=1+\frac{1}{x^2}+\frac{1+\frac{1}{x^2}}{2F_1(x)-1}-\frac{2f_1(x)(x-\frac{1}{x})}{(2F_1(x)-1)^2}\\
&>1+h_1(x)(2F_1(x)-1)^2>1,
\end{aligned}$$

and

$$h'_3(x)=\frac{-xf_1(x)(1-F_1(x))+f_1^2(x)}{(1-F_1(x))^2}<1$$

by $g_3(x)>0$. Therefore, $h'(x)=h'_2(x)-h'_3(x)>0$. This, together with $h(x^*_1)=H'_1(x^*_1)=0$, implies that $h(x)=0$ has a unique solution at x^*_1. Hence, x^*_1 is the unique solution for $H'_1(x)=0$ in $(0,\infty)$. □

Theorem 9. *Let $\Omega_0=\{\theta=[a_1,\ldots,a_n]^{\mathrm{T}}:a_j\in[\underline{a},\overline{a}],j=1,\ldots,n\}$. Suppose that $d(k)$ is Gaussian distributed with mean 0 and variance σ^2. Then,*

(i) *For given u_0 and C,*

$$\widetilde{\eta}(C,u_0)=\begin{cases}G_\sigma(C-\overline{a}u_0)/u_0^2, & \text{if } C<(\overline{a}+\underline{a})u_0/2;\\ G_\sigma(C-\underline{a}u_0)/u_0^2, & \text{if } C\geq(\overline{a}+\underline{a})u_0/2.\end{cases} \tag{33}$$

(ii) *For a given $u_0>0$, $\widetilde{\eta}(C,u_0)$ is minimized by $C=(\overline{a}+\underline{a})u_0/2$.*
(iii)

$$\widetilde{\eta}*=G_\sigma(\overline{a}u_*-\underline{a}u_*)/u_*^2, \tag{34}$$

where $u_*=2\sigma x^*_1/(\overline{a}-\underline{a})$, x^*_1 is the unique solution of (32) and $C^*=(\overline{a}+\underline{a})u_*/2$ is optimal in the worst-case sense over $\theta\in\Omega_0$.

Proof. (i) Since $a_j<\overline{a}$ for $j=1,\ldots,n$ and $u_0>0$, $C-a_ju_0>C-\overline{a}u_0$. If $C-\overline{a}u_0>0$, since $G'_\sigma(x)>0, x>0$, we have $G_\sigma(C-a_ju_0)>G_\sigma(C-\overline{a}u_0)$. If $C-a_ju_0<0$, since $G'_\sigma(x)<0, x<0$, we have $G_\sigma(C-a_ju_0)<G_\sigma(C-\overline{a}u_0)$. Otherwise, $C-a_ju_0>0>C-\overline{a}u_0$. In the last case, if $\overline{a}u_0-C<C-a_ju_0$, then $G_\sigma(C-\overline{a}u_0)<G_\sigma(C-a_ju_0)$; and otherwise $G_\sigma(C-\overline{a}u_0)>G_\sigma(C-a_ju_0)$. These cases can be summarized to obtain

$$\begin{aligned}
&\max\{G_\sigma(C-\overline{a}u_0),G_\sigma(C-a_ju_0)\}\\
&\quad=\begin{cases}G_\sigma(C-\overline{a}u_0), & \text{if } C<(\overline{a}+a_j)u_0/2;\\ G_\sigma(C-a_ju_0), & \text{if } C\geq(\overline{a}+a_j)u_0/2\end{cases}
\end{aligned}$$

for all $j=1,\ldots,n$, which implies (33) after division by u_0^2.

(ii) In the case of $C<(\overline{a}+a_j)u_0/2$, we have $C-\overline{a}u_0<0$. Since $G'_\sigma(x)<0$ for $x<0$, $G_\sigma(C-\overline{a}u_0)$ is minimized when $2C-(\overline{a}+\underline{a})u_0=0$. In other words,

$$\inf_{C<(\overline{a}+\underline{a})u_0/2}\widetilde{\eta}(C,u_0)=\widetilde{\eta}((\overline{a}+\underline{a})u_0/2,u_0).$$

The case for $C\geq(\overline{a}+\underline{a})u_0/2$ can be proved similarly.

(iii) By (ii), we have

$$\begin{aligned}
\eta^*&=\inf_{u_0>0}G_\sigma((\overline{a}-\underline{a})u_0/2)/u_0^2\\
&=\left(\frac{\overline{a}-\underline{a}}{2}\right)^2\inf_{u_0}H_\sigma\left(\frac{\overline{a}-\underline{a}}{2}u_0\right)
\end{aligned}$$

which is minimized at $u_0=u_*$. □

For example, suppose that $n=1$ and the disturbance of $d(k)$ is Gaussian distributed with zero mean and variance $\sigma^2=625$. Now, suppose that $a\in\Omega=[10,15]$. Then, for given C and u_0, we calculate

$$\widetilde{\eta}(C,u_0)=\max_{a\in[10,15]}G(C-au_0)/u_0^2.$$

For different u_0, $\widetilde{\eta}(C,u_0)$ are plotted in the top plot of Fig. 1. By Theorem 9, for each given u_0, the optimal C is $(\overline{a}+\underline{a})u_0/2$. The plot with optimal C is shown in the bottom of Fig. 1. It is shown that approximately optimal input and threshold are $u_*=19.7$ and $C=118$, respectively.

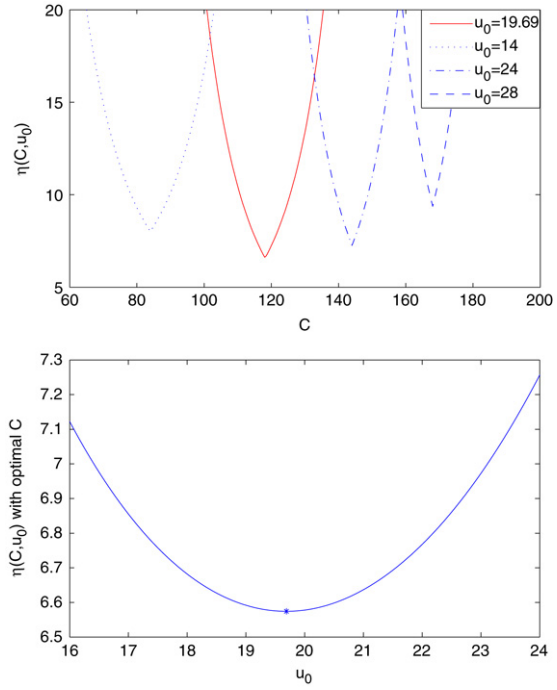

Fig. 1. Optimal worst-case threshold selection and input design for $a \in [10, 15]$ with a Gaussian distributed disturbance of zero mean and variance 625.

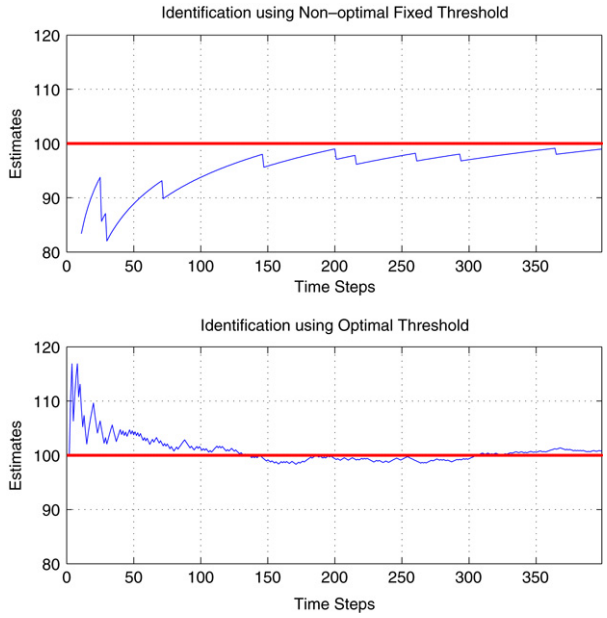

Fig. 2. Comparison of identification accuracy under non-optimal and optimal thresholds: top plot: estimates when a non-optimal threshold ($C = 50$) is used. Bottom plot: estimates when an optimal threshold ($C = 100$) is used.

5.3. Remarks on adaptive design

In the special case of Theorem 7, when the range of uncertainty $\bar{a} - \underline{a}$ is large such that

$$(\bar{a} - \underline{a})u_0 > 2\delta$$

there exists no robustly feasible threshold. In other words, for any threshold $C \in \mathbb{R}$, there exists $\theta \in \Omega$ for which C is not a feasible threshold. In this case, the threshold C must be adaptively selected when more information on θ can be extracted from output observations. More generally, adaptive threshold selection is useful even when a robustly feasible threshold can be found since it can potentially further reduce the errors in Theorem 7.

Consider again Example 2. Fig. 2 demonstrates an example of Gaussian distributed noise and the benefit of using the optimal threshold on enhancing convergence speed. The top plot indicates the estimate trajectory when a non-optimal threshold $C = 50$ is used. The bottom plot shows the estimates when the threshold is optimally selected with $C = 100$. A better convergence speed can be expected when the optimal threshold is used.

Conceptually, when C is not feasible, either $s(k) \equiv 0$ with probability 1, indicating that C is too small, or $s(k) \equiv 1$ with probability 1, indicating that C is too big. On the other hand, when C is feasible and θ is known, Theorem 9 confirms that the optimal threshold is $C = (\bar{a} + \underline{a})u_0/2$ for each given u_0.

When θ is unknown, since x_1^* in (32) can be calculated off-line, a potential adaptive threshold selection algorithm is simply

$$C(N) = (\bar{a}(N) + \underline{a}(N))u_0/2 \tag{35}$$

on the basis of estimated parameter values. There are several potential approaches of adaptation.

The simplest method is a one-step adaptation method: (1) Select an initial threshold C and utilize the initial data set of length N_0 to obtain a parameter estimate $\widehat{\theta}$. (2) The new $\widehat{\theta}$ is then used to update the threshold C to its new value $\widehat{C}$. (3) The remaining data are then used to identify θ with potentially improved identification speed. The main advantage of this approach is its simplicity. Convergence analysis of the early sections remains intact. The loss of data (N_0 points) is compensated by enhanced speed. Selection of N_0 can be analyzed by comparing the benefits of speed improvement and penalty of data loss.

More generally, one may adapt C persistently. While conceptually this approach can potentially improve convergence speed, it introduces further complications in algorithm development and convergence analysis. There are two intervening dynamic processes, one for C adaptation and the other for θ estimation. Convergence analysis of such adaptation schemes is an open and challenging problem. Development of such algorithms is beyond the scope of this paper, and hence is not pursued further.

6. Quantized sensors and optimal resource allocation

Bandwidth resources that limit data-flow rates will be denoted by R in bps. R is related to space and time complexities by $R = N \log(m + 1)$. Suppose that the prior uncertainty set on θ is Ω.

6.1. Optimal resource allocation problems

To understand the impact of increasing m, we revisit

$$\Sigma_0(m, \theta) = N\Sigma_{CR}(m, \theta);$$
$$\varrho(m, \theta) = \max_{\|x\|=1} \|\Sigma_0(m, \theta)x\|$$

in (22). The following two optimal resource allocation problems, being natural dual to each other, are introduced, where $\mathbb{Z}_+$ denotes the set of positive integers.

1. **Optimal uncertainty reduction:** This aims at reducing $\varrho(N, m, \theta)$ for a given resource R.

$$\varepsilon(R) = \min_{m \in \mathbb{Z}_+} \max_{\theta \in \Omega} \varrho(m, \theta)/N,$$
$$\text{subject to } N \log(m + 1) \le R. \tag{36}$$

2. **Optimal resource allocation:** This aims at reducing R for a given error tolerance level ε, i.e., $\varrho(m, \theta)/N \le \varepsilon$.

$$R(\varepsilon) = \min_{m, N \in \mathbb{Z}_+} N \log(m + 1),$$
$$\text{subject to } \max_{\theta \in \Omega} \varrho(m, \theta)/N \le \varepsilon. \tag{37}$$

We will consider two scenarios of increasing space complexity. (1) Structured thresholds: The sets of thresholds are confined to a pre-specified class that satisfies the following condition. For $m_1 < m_2$, the corresponding threshold sets $\mathcal{S}_{m_1}$ and $\mathcal{S}_{m_2}$ satisfy the ordered refinement condition: $\mathcal{S}_{m_1} \subset \mathcal{S}_{m_2}$. For instance, in the typical situation of quantization, one may start with a level of quantization. Then space complexity is increased by sub-dividing each subset $[C_j, C_{j+1})$ by 2 (an increase of space complexity by 1 bit). (2) Unstructured thresholds: For a given m, the threshold values in $\mathcal{S}_m = \{C_1, \ldots, C_m\}$ can be arbitrarily selected. This is the case, for example, when the selection of the thresholds is considered part of coding for communications. In this case, the thresholds can be designed to minimize communication resource utility.

To indicate the dependence on the threshold values, rather than only the number of thresholds, we will use the notation

$$\varrho(\mathcal{S}_m, \theta) = \varrho(m, \theta)$$

in the following subsections.

6.2. Resource allocation with structured thresholds

In this scenario, for all m the threshold sets $\mathcal{S}_m$ are fixed and satisfy the monotone refinement structure $\mathcal{S}_{m_1} \subset \mathcal{S}_{m_2}$ whenever $m_1 < m_2$. We have the following monotonicity in terms of space complexity.

Corollary 2. *Under the conditions of Theorem 5,*

$$\varrho(\mathcal{S}_{m_2}, \theta) \le \varrho(\mathcal{S}_{m_1}, \theta).$$

For a given resource $R = N \log(m+1)$, $N = R/\log(m+1)$. As a result, asymptotically

$$\frac{\varrho(\mathcal{S}_m, \theta)}{N} = \frac{\log(m+1)\varrho(\mathcal{S}_m, \theta)}{R}.$$

An optimal resource allocation for the given R is

$$\varepsilon_1(R) = \frac{\min\limits_{1 \le m \le 2^R - 1} \log(m+1)\varrho(\mathcal{S}_m, \theta)}{R}. \tag{38}$$

Example 3. Consider the system $y(k) = \theta + d(k)$. Suppose that the disturbance is Gaussian distributed with zero mean and variance 200. Hence, the probability density function is

$$f(x) = \frac{e^{-\frac{x^2}{400}}}{\sqrt{400\pi}}.$$

The actual value of θ is 55. Thresholds are structured as follows. The interval of thresholds is $[-10, 70]$. Initially, one sensor threshold is placed at $C = 30$ (the middle point of the interval) with space complexity $\log(m+1) = \log 2 = 1$ bit. To increase space complexity, the number m of thresholds is gradually increased by dividing equally the interval $[-10, 70]$. Fig. 3 shows $\varrho(\mathcal{S}_m, \theta)$ and $\log(m+1)\varrho(\mathcal{S}_m, \theta)$ as functions of the space complexity m. For this example, the optimal space complexity is $m = 3$ thresholds.

The space complexity depends on the actual values of θ and threshold choices. Its dependence on θ is illustrated in Fig. 4 in which the space complexities for three different θ values are plotted.

Furthermore, the space complexity varies significantly with placement of the thresholds. Fig. 5 shows the space complexity when the range of thresholds is changed from $[-10, 70]$ to $[-10, 60]$. The optimal number of thresholds becomes $m = 2$.

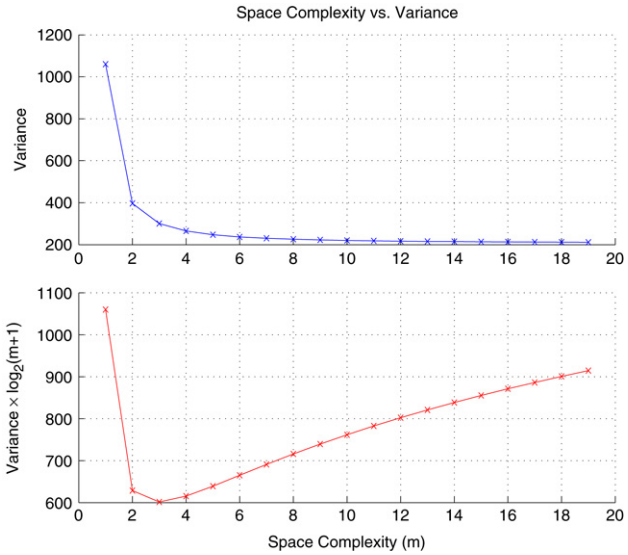

Fig. 3. Space complexity: $\varrho(\mathcal{S}_m, \theta)$ vs. $\log(m+1)$ (top plot); $\log(m+1)\varrho(\mathcal{S}_m, \theta)$ vs. $\log(m+1)$ (bottom plot).

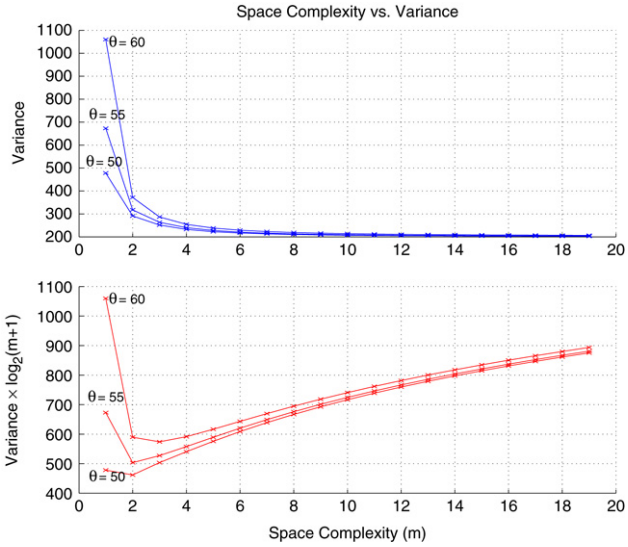

Fig. 4. Different space complexity curves: $\theta = 50, 55, 60$.

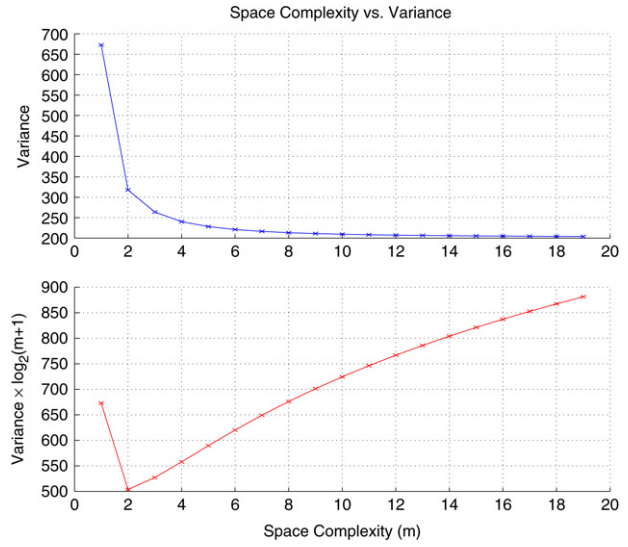

Fig. 5. Space complexity curve varies with thresholds: $\theta = 55$.

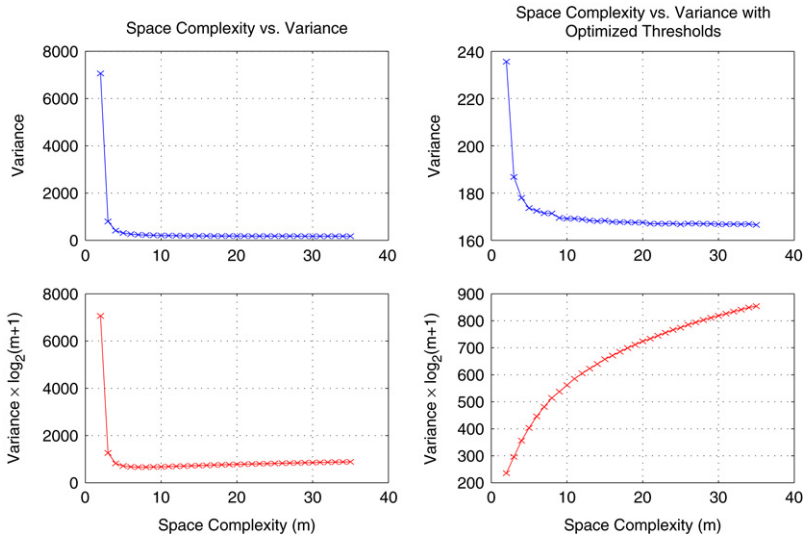

Fig. 6. Comparison of space complexity: (i) Left plots: structured thresholds; (ii) Right plots: optimized thresholds. $\theta = 55$.

6.3. Resource allocation with unstructured thresholds

When the thresholds are design variables, the space complexity is defined as follows. Let

$$\mathbf{S}_m = \{\mathcal{S}_m = \{C_1, \ldots, C_m\} : y_{\min} < C_1 \le C_2 \le \cdots \le C_m < y_{\max}, j = 1, \ldots, m\}. \quad (39)$$

This is the set of all possible m thresholds. Noticing that in (39), we allow thresholds to be repeated. Consequently, for any $\mathcal{S}_m \in \mathbf{S}_m$, there exists (infinitely many) $\mathcal{S}_{m+1} \in \mathbf{S}_{m+1}$ such that $\mathcal{S}_{m+1}$ is a refinement of $\mathcal{S}_m$.

Definition 3. $\varrho_m(\theta) = \inf_{\mathcal{S}_m \in \mathbf{S}_m} \varrho(\mathcal{S}_m, \theta)$.

By Theorem 5, we have the following monotonicity in terms of space complexity.

Corollary 3. *If* $m_1 < m_2$, *then*

$$\varrho_{m_2}(\theta) \le \varrho_{m_1}(\theta).$$

Proof. For any $\mathcal{S}_{m_1} \in \mathbf{S}_{m_1}$, there exists an $\mathcal{S}_{m_2} \in \mathbf{S}_{m_2}$ such that $\mathcal{S}_{m_2}$ is a refinement of $\mathcal{S}_{m_1}$. By Theorem 5, we have

$$\varrho(\mathcal{S}_{m_2}, \theta) \le \varrho(\mathcal{S}_{m_1}, \theta).$$

Consequently,

$$\varrho_{m_2}(\theta) = \inf_{\mathcal{S}_{m_2} \in \mathbf{S}_{m_2}} \varrho(\mathcal{S}_{m_2}, \theta) \le \inf_{\mathcal{S}_{m_1} \in \mathbf{S}_{m_1}} \varrho(\mathcal{S}_{m_1}, \theta) = \varrho_{m_1}(\theta). \quad \square$$

For a given resource $R = N \log(m+1)$, we have $N = R/\log(m+1)$. As a result, asymptotically

$$\sigma^2_{CR}(N, m, \theta) = \frac{\varrho_m(\theta)}{N} = \frac{\log(m+1)\varrho_m(\theta)}{R}.$$

An optimal resource allocation for the given R is

$$\varepsilon_2(R) = \frac{\min_{1 \le m \le 2^R - 1} \log(m+1)\varrho_m(\theta)}{R}. \quad (40)$$

Example 4. Consider the same system setting as in Example 3. Suppose that the disturbance is Gaussian distributed with zero mean and variance 150. The true value of $\theta = 55$. Two scenarios are compared: (1) Thresholds are structured. The interval of thresholds is $[-20, 60]$. Initially, one sensor threshold is placed at $C = 20$, with space complexity $\log(m+1) = \log 2 = 1$ bit. To increase space complexity, the number of thresholds is increased, each time the range is divided equally. (2) Thresholds are optimized for maximum reduction of variances. Fig. 6 demonstrates the benefit of choosing optimal thresholds for complexity reduction. The plots show that (1) Optimization of thresholds can greatly reduce identification errors; (2) Optimal space complexity can be greatly reduced. For the structured thresholds, the optimal space complexity is $m = 6$ thresholds. For optimized threshold selection, it becomes 1 bit, i.e., a binary sensor is the optimal choice in terms of space complexity. The plots show that also the optimized thresholds reduce variance significantly (663 vs. 236 at the respective optimal space complexities).

6.4. Discussions on space and time complexity

The above examples highlight a number of interesting facts about space and time complexities.

(1) It is observed that initial increase of space complexity induces a sharp drop in variance. However, variances soon reach a near-constant level that does not reduce significantly with increased space complexity. The optimal space complexities in these examples are surprisingly low, $m = 3$ in structured thresholds and $m = 1$ in unstructured thresholds. It implies that when observations are corrupted by random noises, much more resources should be devoted to heightening data size, rather than data precision.

(2) A simple calculation shows that resource allocation is a significant issue in identification problems. A common quantization scheme in data processing will carry B bits precision. Take an example of $B = 10$, namely $m = 2^{10} - 1 = 1023$ thresholds. From Fig. 3, $\varrho(\mathcal{S}_{1023}, \theta)$ approaches a constant about 200 for large m values. To reduce the variance to, say, 0.1, the observation length N must be larger than $N \ge 200/0.1 = 2000$. Together, this amounts to $R_0 \ge NB = 20$ K bits resource. For a rational system containing 20 parameters, the total resource will be $R = 20R_0 = 400$ K bits. Optimal resource allocations discussed in this paper indicate that this resource request can

be greatly reduced if one chooses wisely space complexity. For this example, from Fig. 3 one may choose $m = 3$ as the optimal space complexity. To achieve the same variance of 0.1, we only need $R_0 = 600/0.1 = 6$ K bits resource, a large reduction from 20 K bits.

(3) Space complexity depends on the actual parameters. However, from Fig. 4 it seems that optimal space complexity is always low.

7. Conclusions

Tradeoff between space complexity (measurement precision) and time complexity (observation length) is of fundamental importance in system modeling, identification, and information processing when information processing speed and data-flow rates are limited. The issue is inherent in all problems involving signal digitization (sampling and quantization), but most relevant in systems involving communications, wireless connections, or computer networks. This paper introduces a basic framework and certain essential tools for analyzing space and time complexities, delineating the tradeoff in terms of identification accuracy, and optimizing resource utility.

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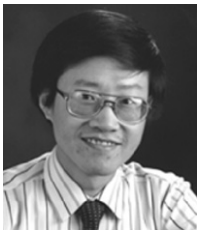

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